

Seymour's Second Neighborhood Conjecture for orientations of (pseudo)random graphs

Fábio Botler¹ Phablo F. S. Moura² Tássio Naia³

June 19, 2025

¹ Programa de Engenharia de Sistemas e Computação
Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa em Engenharia
Universidade Federal do Rio de Janeiro, Brasil
`fbotler@cos.ufrj.br`

² Research Center for Operations Research & Statistics,
KU Leuven, Belgium
`phablo.moura@kuleuven.be`

³ Departamento de Ciência da Computação
Instituto de Matemática e Estatística
Universidade de São Paulo, Brasil
`tnaia@member.fsf.org`

Abstract

Seymour's Second Neighborhood Conjecture (SNC) states that every oriented graph contains a vertex whose second neighborhood is as large as its first neighborhood. We investigate the SNC for orientations of both binomial and pseudo random graphs, verifying the SNC asymptotically almost surely (a.a.s.)

- (i) for all orientations of $G(n, p)$ if $\limsup_{n \rightarrow \infty} p < 1/4$; and
- (ii) for a uniformly-random orientation of each weakly $(p, A\sqrt{np})$ -bijumbled graph of order n and density p , where $p = \Omega(n^{-1/2})$ and $1 - p = \Omega(n^{-1/6})$ and $A > 0$ is a universal constant independent of both n and p .

We also show that a.a.s. the SNC holds for almost every orientation of $G(n, p)$. More specifically, we prove that a.a.s.

- (iii) for all $\varepsilon > 0$ and $p = p(n)$ with $\limsup_{n \rightarrow \infty} p \leq 2/3 - \varepsilon$, every orientation of $G(n, p)$ with minimum outdegree $\Omega_\varepsilon(\sqrt{n})$ satisfies the SNC; and
- (iv) for all $p = p(n)$, a random orientation of $G(n, p)$ satisfies the SNC.

We remark that either (iii) or (iv) confirms the SNC for almost every oriented graph.

1 Introduction

An *oriented graph* D is a directed graph (digraph) obtained from a simple graph G by assigning directions to its edges (i.e., D contains neither loops, nor parallel arcs,

nor directed cycles of length 2); we also call D an *orientation* of G . Given $i \in \mathbb{N}$, the i -th neighborhood of $u \in V(D)$, denoted by $N^i(u)$, is the set of vertices v for which a shortest directed path from u to v has precisely i arcs. A *Seymour vertex* (see [14]) is a vertex u for which $|N^2(u)| \geq |N^1(u)|$. Seymour conjectured the following (see [6]).

Conjecture 1. Every oriented graph contains a Seymour vertex.

Conjecture 1, known as *Seymour's Second Neighborhood Conjecture* (SNC), is a notorious open question (see, e.g., [3, 7, 9, 14]). In particular, it was confirmed for tournaments (orientations of cliques) by Fisher [8] and (with a purely combinatorial argument) by Havet and Thomassé [10]; it was also studied by Cohn, Godbole, Harkness and Zhang [4] for the random digraph model in which each ordered pair of vertices is picked independently as an arc with probability $p < 1/2$.

Our contribution comes from considering this combinatorial problem in a random and pseudorandom setting (see, e.g., [5, 13]). More precisely, we explore Conjecture 1 for orientations of the binomial random graph $G(n, p)$, defined as the random graph with vertex set $\{1, \dots, n\}$ in which every pair of vertices appears as an edge independently and with probability p .

In this paper, we denote by \mathcal{S} the set of graphs G such that every orientation of G contains a Seymour vertex. We say that an event \mathcal{E} holds *asymptotically almost surely* (a.a.s.) if $\Pr[\mathcal{E}] \rightarrow 1$ as $n \rightarrow \infty$. If $G = G(n, p)$ is very sparse (say, if $np \leq (1 - \varepsilon) \ln n$ for large n and fixed $\varepsilon > 0$), then a.a.s. G has an isolated vertex, which clearly is a Seymour vertex. Our first result extends this observation to much denser random graphs.

Theorem 2. Let $p: \mathbb{N} \rightarrow (0, 1)$. If $\limsup_{n \rightarrow \infty} p < 1/4$, then a.a.s. $G(n, p) \in \mathcal{S}$.

If we impose restrictions on the orientations, requiring, for example, somewhat large minimum outdegree, the range of p can be further increased.

Theorem 3. For every $\beta > 0$, there exists $C = C(\beta)$ such that the following holds for all $p: \mathbb{N} \rightarrow (0, 1)$. If $\limsup_{n \rightarrow \infty} p \leq 2/3 - \beta$, then a.a.s. every orientation of $G(n, p)$ with minimum outdegree at least $Cn^{1/2}$ contains a Seymour vertex.

In general, for $p \in (0, 1)$, we show that a.a.s. *most* orientations of $G(n, p)$ contain a Seymour vertex. In particular, either Theorem 3 or the following result imply that Conjecture 1 holds for almost every labeled oriented graph.

Theorem 4. Let $p: \mathbb{N} \rightarrow (0, 1)$ and let $G = G(n, p)$. If D is chosen uniformly at random among the $2^{e(G)}$ orientations of G , then a.a.s. D has a Seymour vertex.

In fact, we prove a version of Theorem 4 in a more general setting, namely orientations of pseudorandom graphs (see Section 4).

Theorem 5. For every $A > 0$, there exists a constant $C = C(A) > 1$ such that the following holds for every $p, \varepsilon \in (0, 1)$ and $n \in \mathbb{N}$ satisfying $\varepsilon^3 np^2 \geq A^2 C$ and $p < 1 - 13\sqrt{\varepsilon}$. Let G be a weakly $(p, A\sqrt{np})$ -bijumbled graph of order n . If D is chosen uniformly at random among the $2^{e(G)}$ possible orientations of G , then D has a Seymour vertex with probability at least $1 - 2^{-n}$.

This paper is organized as follows. In Section 2 we prove Conjecture 1 for wheel-free graphs, which implies the particular case of Theorem 2 when $n^2 p^3 \rightarrow 0$. In Section 3 we complete the proof of Theorem 2 and prove Theorems 3 and 4 using a set of standard properties of $G(n, p)$. These properties are collected in Definition 9 and Lemma 10 (proved in Appendix A). In Section 4, we introduce bijumbled graphs and prove Theorem 5. We make a few further remarks in Section 5.

To avoid uninteresting technicalities, we omit floor and ceiling signs. The asymptotic expressions $O(\cdot)$, $\Omega(\cdot)$, $o(\cdot)$ are relative to $n \rightarrow \infty$, and $x = \Omega_\varepsilon(f(n))$ means that there exist constants n_0 and C depending only on ε such that $x \geq Cf(n)$ for all $n \geq n_0$. If S and T are sets of vertices in a graph (or oriented graph) H , we denote by $\vec{e}_H(S, T)$ the number of arcs directed from S to T , by $e_H(S, T)$ the number of edges or arcs with one vertex in each set, and by $e_H(S)$ the number of edges or arcs with both vertices in S . We emphasize that edges in $S \cap T$ are counted twice in $e_H(S, T)$. On the other hand, $\vec{e}(S, T)$ counts arcs induced by $S \cap T$ only once. Also, as usual, $H[S]$ denotes the (oriented) graph induced by the vertices in S .

Given a vertex u in an oriented graph H , we denote by $\deg_H^+(u)$ the *outdegree* of u in H , i.e., the number of arcs of H leaving u ; we denote by $\delta^+(H)$ the minimum outdegree over all vertices of H . The (underlying) *neighborhood* of a vertex u is denoted by $N_H(u)$, and the *codegree* of vertices u, v is $\deg_H(u, v) = |N_H(u) \cap N_H(v)|$. Subscripts are omitted when the intended (oriented) graph is clear from context.

Theorem 2 and a weaker version of Theorem 3 appeared in the extended abstracts [1, 2].

2 Wheel-free graphs

A *wheel* is a graph obtained from a cycle C by adding a new vertex adjacent to all vertices in C . Firstly, we show that $G(n, p)$ is wheel-free when p is small; then prove that all wheel-free graphs satisfy Conjecture 1.

Lemma 6. If $p \in (0, 1)$ and $n^4 p^6 < \varepsilon/16$, then $\Pr[G(n, p) \text{ is wheel-free}] \geq 1 - \varepsilon$.

Proof. We can assume $\varepsilon < 1$. Since $n^4 p^6 < \varepsilon/16$, we have that

$$np^2 < (\varepsilon p^2/16)^{1/4} < 1/2. \quad (1)$$

Let $X = \sum_{k=4}^n X_k$, where X_k denotes the number of wheels of order k in $G(n, p)$. By

the linearity of expectation,

$$\begin{aligned} \mathbb{E} X &= \sum_{k=4}^n \mathbb{E} X_k = \sum_{k=4}^n \binom{n}{k} k \frac{(k-1)!}{2(k-1)} p^{2(k-1)} \\ &< n \sum_{k=4}^n (np^2)^{k-1} = n^4 p^6 \sum_{k=0}^{n-4} (np^2)^k \stackrel{\text{G.S.}}{<} \frac{n^4 p^6}{1 - np^2} \stackrel{(1)}{<} 2n^4 p^6 < \frac{\varepsilon}{8} < \varepsilon, \end{aligned} \quad (2)$$

where in (2) we use the formula $\sum_{i=0}^{\infty} r^i = (1-r)^{-1}$ for the geometric series (G.S.) of ratio $r = np^2 < 1$. Markov's inequality then yields $\Pr[X \geq 1] \leq \mathbb{E} X < \varepsilon$. \square

To show that every orientation of a wheel-free graph has a Seymour vertex, we prove a slightly stronger result. A digraph is *locally cornering* if the outneighborhood of each vertex induces a digraph with a *sink* (i.e., a vertex of outdegree 0). Note that any vertex of minimum outdegree is a Seymour vertex in a locally cornering oriented graph.

Proposition 7. Every locally cornering oriented graph has a Seymour vertex. \square

Lemma 6 and Proposition 7 immediately yield the following corollary.

Corollary 8. If $p \in (0, 1)$, and $n^4 p^6 < \varepsilon/16$, then $\Pr[G(n, p) \in \mathcal{S}] \geq 1 - \varepsilon$. \square

Proof. Note that every orientation of a wheel-free graph is locally cornering, since the (out)neighborhood of each vertex is a forest, and every oriented forest has a vertex with outdegree 0. Hence the result follows by Lemma 6 and Proposition 7. \square

3 Typical graphs

In this section we prove that if $\limsup_{n \rightarrow \infty} p < 1/4$, then a.a.s. $G(n, p) \in \mathcal{S}$. We use a number of standard properties of $G(n, p)$, stated for convenience in Definition 9.

Definition 9. Let $p \in (0, 1)$. A graph G of order n is *p-typical* if the following hold.

(i) For every $X \subseteq V(G)$, we have

$$\left| e(X) - \binom{|X|}{2} p \right| \leq |X| \sqrt{3np(1-p)} + 2n.$$

(ii) If $n' \ln n \leq n'' \leq n$ or $n' = n'' = n$, then all disjoint $X, Y \subseteq V(G)$ with $|X|, |Y| \leq n'$ satisfy

$$|e(X, Y) - |X||Y|p| \leq \sqrt{6n''p(1-p)|X||Y|} + 2n''.$$

(iii) For every $v \in V(G)$, we have

$$|\deg(v) - (n-1)p| \leq \sqrt{6np(1-p) \ln n} + 2 \ln n.$$

(iv) For every distinct $u, v \in V(G)$, we have

$$|\deg(u, v) - (n-2)p^2| \leq \sqrt{6np^2(1-p^2) \ln n} + 2 \ln n.$$

It can be shown, using standard Chernoff-type concentration inequalities, that $G(n, p)$ is p -typical with high probability (see Appendix A).

Lemma 10. For every $p: \mathbb{N} \rightarrow (0, 1)$, a.a.s. $G = G(n, p)$ is p -typical.

We also use the following property of graphs satisfying Definition 9 (i).

Lemma 11. Let G be a graph of order n which satisfies Definition 9 (i), and fix $a \in \mathbb{N}$. If D is an orientation of G and $B = \{v \in V(D) : \deg_D^+(v) < a\}$, then

$$|B| \leq \frac{2}{p}(a-1) + 1 + \sqrt{\frac{12n(1-p)}{p}} + \frac{4n}{|B|p}.$$

Proof. The lemma follows by multiplying all terms in the inequality below by $2/|B|p$.

$$|B|(a-1) \geq e(G[B]) \stackrel{9(i)}{\geq} \binom{|B|}{2} p - |B|\sqrt{3np(1-p)} - 2n. \quad \square$$

3.1 Proof of Theorem 2

Let us outline the proof of Theorem 2. Recall that, by Corollary 8, we can assume $n^4 p^6 = \Omega(1)$. Firstly, we find a vertex w whose outneighborhood contains many vertices with large outdegree. Then, we note that $|N^1(w)| = \Theta(np)$ and that the bipartite graph consisting of the edges joining vertices in $N^1(w)$ to vertices in $N^2(w)$ cannot be too dense. Finally, since many outneighbors of w have large outdegree, we conclude that $N^1(w) \cup N^2(w)$ must contain at least $2|N^1(w)|$ vertices, completing the proof. This yields the following.

Lemma 12. Fix $0 < \alpha < 1/4$ and $\varepsilon \in (0, 1)$. There is $n_1 = n_1(\alpha, \varepsilon)$ such that \mathcal{S} contains all p -typical graphs of order n such that $n \geq n_1$ and $\varepsilon n^{-2/3} \leq p \leq 1/4 - \alpha$.

Lemma 12 is our last ingredient for proving Theorem 2. Indeed, fix $\varepsilon \in (0, 1)$, set $\alpha = 1/4 - \limsup_{n \rightarrow \infty} p(n)$ and let n_0 be large enough so that $p(n) \leq 1/4 - \alpha$ and so that $G(n, p)$ is p -typical with probability at least $1 - \varepsilon$ for all $n \geq n_0$ (this is Lemma 10). Now either $p < \varepsilon n^{-2/3}$ or $\varepsilon n^{-2/3} \leq p(n) \leq 1/4 - \alpha$. In the former case we use Corollary 8, and in the latter case Lemma 12, concluding either way that

$$\Pr[G(n, p) \in \mathcal{S}] \geq 1 - \varepsilon.$$

Proof of Lemma 12. We may and shall assume (choosing n_1 accordingly) that np is large enough whenever necessary. Now, let G be a p -typical graph of order n , and fix an arbitrary orientation of G . For simplicity, we write G for both the oriented and underlying graphs. Let

$$S = \{v \in V(G) : \deg^+(v) < (1 - \alpha)np/2\}$$

and $T = V(G) \setminus S$. Firstly, we show that $|T| \geq \alpha n/2$. This is clearly the case if $|S| < \alpha n$ (since $\alpha < 1/4 < 1 - \alpha$); let us show that this also holds if $|S| \geq \alpha n$. Indeed,

since $p \geq \varepsilon n^{-2/3}$, from Lemma 11 with $a = (1 - \alpha)np/2$ we obtain

$$|S| \leq \frac{2(a-1)}{p} + 1 + \sqrt{\frac{12n(1-p)}{p}} + \frac{4n}{|S|p} = (1 - \alpha)n + o(n) < \left(1 - \frac{\alpha}{2}\right)n.$$

Therefore $|T| = n - |S| \geq \alpha n/2$ as desired. Recall that np is large and $p \leq 1/4$. Then $\sqrt{3np(1-p)} \geq 4/\alpha$, and hence, from Definition 9 (i), we get

$$e(T) \geq \binom{|T|}{2}p - |T|\sqrt{3np(1-p)} - 2n > \binom{|T|}{2}p - 2|T|\sqrt{np} > \frac{|T|^2 p}{3}, \quad (3)$$

and therefore, by averaging, there exists $w \in T$ satisfying

$$\deg_{G[T]}^+(w) \geq \frac{e(T)}{|T|} \stackrel{(3)}{\geq} \frac{\alpha np}{6}. \quad (4)$$

We next show that w is a Seymour vertex. Let $X = N_G^1(w)$ and $Y = N_G^2(w)$, and suppose, for a contradiction, that $|Y| < |X|$. From Definition 9 (iii) and $p + \alpha \leq 1/4$, we have

$$|X| \leq np + \sqrt{6np \ln n} + 2 \ln n < n \left(p + \frac{\alpha}{2}\right) < \frac{n}{4} \leq \frac{n}{2}(1 - 2\alpha - 2p). \quad (5)$$

Moreover,

$$|X| \leq np + \sqrt{6np \ln n} + 2 \ln n < 2np. \quad (6)$$

Recall that $w \in T$ and let $N = X \cap T$ be the set of outneighbors of w in T . By the definition of N and (4) we have

$$|N| \geq \frac{\alpha np}{6}. \quad (7)$$

Recall that $\vec{e}(N, X)$ counts arcs induced by N precisely once (as $N \subseteq X$), and if the arc $u \rightarrow v$ is counted by $\vec{e}(N, X)$, then v is a common neighbor of w and $u \in N$. Hence, by Definition 9 (iv), we have that

$$\vec{e}(N, X) \leq \sum_{u \in N} \deg(w, u) \leq |N|(np^2 + \sqrt{6np^2 \ln n} + 2 \ln n). \quad (8)$$

Since vertices in T (and hence in N) have at least $(1 - \alpha)np/2$ outneighbors, we have

$$\begin{aligned} \vec{e}(N, Y) &\geq |N| \frac{(1 - \alpha)np}{2} - \vec{e}(N, X) \\ &\stackrel{(8)}{\geq} |N| \frac{(1 - \alpha)np}{2} - |N|(np^2 + \sqrt{6np^2 \ln n} + 2 \ln n). \end{aligned} \quad (9)$$

The following estimate will be useful.

Claim 13. It holds that $2 \ln n + \sqrt{6np^2 \ln n} + \sqrt{6|Y|np/|N|} = o(np)$.

Proof. We prove that each term in the sum above is $o(n)$ when divided by p . Clearly, $\sqrt{6np^2 \ln n}/p = o(n)$. Recall that $p \geq \varepsilon n^{-2/3}$ and thus $(2 \ln n)/p = o(n)$. Also,

$$\sqrt{\frac{|Y|6n}{|N|p}} \stackrel{(7)}{\leq} \sqrt{\frac{|Y|36}{\alpha p^2}} < \sqrt{\frac{|X|36}{\alpha p^2}} \stackrel{(6)}{<} \sqrt{\frac{72n}{\alpha p}} = o(n). \quad \diamond$$

We divide the remainder of the proof into two cases. Fix $\gamma \in (1/2, 2/3)$.

163 **Case 1.** Suppose firstly that $p > n^{\gamma-1}/2$. Since $N \cap Y = \emptyset$, Definition 9 (ii) yields

$$164 \quad \bar{e}(N, Y) \leq e(N, Y) \leq |N||Y|p + \sqrt{6np|N||Y|} + 2n. \quad (10)$$

165 Thus, combining (9) and (10), we have

$$166 \quad \frac{(1-\alpha)np}{2} - (np^2 + \sqrt{6np^2 \ln n} + 2 \ln n) \leq |Y|p + \sqrt{\frac{6np|Y|}{|N|}} + \frac{2n}{|N|}. \quad (11)$$

167 Also note that since $p > n^{\gamma-1}/2$ and $\gamma > 1/2$, we can estimate

$$168 \quad \frac{2n}{|N|p} \stackrel{(7)}{\leq} \frac{12}{\alpha p^2} < \frac{24}{\alpha n^{2\gamma-2}} = o(n). \quad (12)$$

169 Finally, we conclude that w is a Seymour vertex, since (11) becomes

$$170 \quad |Y| \geq \frac{(1-\alpha-2p)n}{2} - \sqrt{6n \ln n} - \sqrt{\frac{6n|Y|}{|N|p}} - \frac{2n}{|N|p} - \frac{2 \ln n}{p}$$

$$171 \quad \stackrel{(*)}{\geq} \frac{n}{2} (1-2\alpha-2p) \stackrel{(5)}{>} |X|,$$

172 where inequality $(*)$ follows from Claim 13 and (12).

173 **Case 2.** Suppose now that $p \leq n^{\gamma-1}/2$. In this case (6) implies $|X| \leq n^\gamma$. Since
 174 $N \subseteq X$, $N \cap Y = \emptyset$, and $|Y| < |X|$, Definition 9 (ii) (with $n' = n^\gamma$ and $n'' = n^\gamma \ln n$)
 175 yields

$$176 \quad \bar{e}(N, Y) \leq |N||Y|p + \sqrt{6(n^\gamma \ln n)p|N||Y|} + 2n^\gamma \ln n$$

$$177 \quad < |N||Y|p + \sqrt{6np|N||Y|} + 2n^\gamma \ln n. \quad (13)$$

178 Now, from (9) and (13), we obtain the following inequality, which is analogous
 179 to (11), but with the term $2n/|N|$ replaced by $2n^\gamma \ln n/|N|$.

$$180 \quad \frac{(1-\alpha)np}{2} - (p^2n + \sqrt{6np^2 \ln n} + 2 \ln n) \leq |Y|p + \sqrt{\frac{6np|Y|}{|N|}} + \frac{2n^\gamma \ln n}{|N|}. \quad (14)$$

181 We claim that $2n^\gamma \ln n/|N| = o(np)$. Indeed, since $p \geq \varepsilon n^{-2/3}$ and $\gamma < 2/3$, we have

$$182 \quad \frac{2n^\gamma \ln n}{|N|p} \stackrel{(7)}{\leq} \frac{12n^\gamma \ln n}{\alpha np^2} = \frac{12n^{\gamma-1} \ln n}{\alpha p^2} \leq \frac{12n^{\gamma+1/3} \ln n}{\alpha \varepsilon^2} = o(n). \quad (15)$$

183 We complete the proof of Case 2 by solving (14) for $|Y|$ as in Case 1 (using Claim 13
 184 and (15) to estimate $2n^\gamma \ln n/|N|$). \square

185 3.2 Proof of Theorem 4

186 We are now in a position to prove Theorem 4, which we restate for convenience.

Theorem 4. Let $p: \mathbb{N} \rightarrow (0, 1)$, and let $G = G(n, p)$. If D is chosen uniformly at random among the $2^{e(G)}$ orientations of G , then a.a.s. D has a Seymour vertex.

187 *Proof of Theorem 4.* Let $G = G(n, p)$. If $p < 1/5$, then $\Pr[G \in \mathcal{S}] = 1 - o(1)$ by
188 Theorem 2. On the other hand, if $p \geq 1/5$, then standard concentration results for
189 binomial random variables (e.g., Chernoff-type bounds) yield that every ordered
190 pair (u, v) of distinct vertices of G satisfies, say $\deg(u, v) \geq n/50$, and hence with
191 probability $1 - o(1)$ every such pair is joined by a directed path of length 2. This
192 is because building a random orientation of $G(n, p)$ is equivalent to first choosing
193 which edges are present and then choosing the orientation of each edge uniformly
194 at random, with choices mutually independent for each edge. In other words, with
195 probability $1 - o(1)$, for all $u \in V(G)$ we have $V(G) = \{u\} \cup N^1(u) \cup N^2(u)$. Finally, by
196 averaging outdegrees, we can find a vertex $z \in V(D)$ with outdegree at most $(n-1)/2$,
197 because $\sum_{v \in V(D)} \deg_D^+(v) = e(D) \leq n(n-1)/2$. Such z is a Seymour vertex as
198 desired. \square

199 3.3 Orientations with large minimum outdegree

200 Our last result in this section yields yet another class of orientations of p -typical
201 graphs which must always contain a Seymour vertex. In fact, we consider a larger
202 class of underlying graphs, showing that if a graph G satisfies items (i) and (ii) of
203 Definition 9, then every orientation D of G with minimum outdegree $\delta^+(D) = \Omega(n^{1/2})$
204 contains a Seymour vertex. This may be useful towards extending the range of p for
205 which a.a.s. $G(n, p) \in \mathcal{S}$.

Lemma 14. Fix $\beta > 0$. There exist a constant $C = C(\beta)$ and $n_0 = n_0(\beta)$ such
that the following holds for all $n \geq n_0$ and $p \leq 2/3 - \beta$. If G is a graph of order n
that satisfies items (i) and (ii) of Definition 9, then every orientation D of G for
which $\delta^+(D) \geq Cn^{1/2}$ has a Seymour vertex.

206 Note that Lemma 14 and Lemma 10 immediately imply Theorem 3.

207 *Proof of Lemma 14.* Since $(1 - 3p/2) \geq 3\beta/2$, we may fix $C \geq 4$ so that

$$208 \left(1 - \frac{3p}{2}\right)C - \left(\sqrt{3p(1-p)} + \sqrt{6p(1-p)}\right) \geq \frac{3\beta C}{2} - 4 \geq 1.$$

209 Fix $v \in V(D)$ with $\deg_D^+(v) = \delta^+(D)$, let $X = N^1(v)$ and $Y = N^2(v)$. We shall
210 prove that $|X| \leq |Y|$. Suppose to the contrary that $|Y| < |X|$. By Definition 9 (i),

$$211 \begin{aligned} \vec{e}(X, Y) &= \sum_{a \in X} \deg^+(a) - e(X) \geq |X|^2 - \left(\frac{|X|^2 p}{2} + |X| \sqrt{3np(1-p)} + 2n\right) \\ 212 &= \left(1 - \frac{p}{2}\right) |X|^2 - \left(|X| \sqrt{3np(1-p)} + 2n\right), \end{aligned} \quad (16)$$

213 and by Definition 9 (ii) (with $n' = n'' = n$) we have

$$214 \begin{aligned} \vec{e}(X, Y) &\leq e(X, Y) \leq |X||Y|p + \sqrt{6np(1-p)}|X||Y| + 2n \\ 215 &< |X|^2 p + |X| \sqrt{6np(1-p)} + 2n. \end{aligned} \quad (17)$$

216 Since $|X| \geq Cn^{1/2} \geq n^{1/2}$, combining (16) and (17) yields the following contradiction.

$$\begin{aligned}
217 \quad 4n &> \left(1 - \frac{3p}{2}\right)|X|^2 - |X|\left(\sqrt{3np(1-p)} + \sqrt{6np(1-p)}\right) \\
218 \quad &\geq Cn \left(\left(1 - \frac{3p}{2}\right)C - \left(\sqrt{3p(1-p)} + \sqrt{6p(1-p)}\right) \right) \geq 4n. \quad \square
\end{aligned}$$

219 4 Typical orientations of weakly bijumbled graphs

220 In this section, we focus on a well-known class of pseudorandom graphs (that is,
221 deterministic graphs which embody many properties of $G(n, p)$), and argue that
222 almost all of their orientations contain a Seymour vertex. The following results
223 concern graphs of order n and density p , where $Cn^{-1/2} \leq p \leq 1 - \varepsilon$, and $C = C(\varepsilon) > 0$
224 depends only on the constant $\varepsilon > 0$.

Definition 15— (p, α) -**bijumbled**. Let p and α be given. We say that a graph G of order n is *weakly (p, α) -bijumbled* if, for all $U, W \subset V(G)$ with $U \cap W = \emptyset$ and $1 \leq |U| \leq |W| \leq np|U|$, we have

$$|e(U, W) - p|U||W|| \leq \alpha\sqrt{|U||W|}. \quad (18)$$

If (18) holds for all disjoint $U, W \subset V(G)$, then we say that G is *(p, α) -bijumbled*.

225 We note that the random graph is a.a.s. (weakly) bijumbled.

Theorem 16 – **Lemma 3.8** in [11]. For any $p : \mathbb{N} \rightarrow (0, 1]$, the random graph $G(n, p)$ is a.a.s. weakly $(p, A\sqrt{np})$ -bijumbled for a certain absolute constant $A \leq e^2\sqrt{6}$.

226 In what follows, A shall always denote the constant from Theorem 16. A simple
227 double-counting argument shows the following.

Fact 17. If G is weakly (p, α) -bijumbled, then for every $U \subset V(G)$ we have

$$\left| e(G[U]) - p \binom{|U|}{2} \right| \leq \alpha|U|. \quad (19)$$

228 The next lemma lists the remaining properties of weakly bijumbled graphs which
229 we use.

Lemma 18. There exists a universal constant $C > 1$ such that if $A \geq 2$ and $\varepsilon, p \in (0, 1)$ are such that $\varepsilon^3 np^2 \geq A^2 C$, then every weakly $(p, A\sqrt{np})$ -bijumbled graph G of order n satisfies the following properties.

(i) $|\{v \in V(G) : |\deg(v) - np| > \varepsilon np\}| \leq \varepsilon n$.

(ii) For every orientation of G and every integer d , we have

$$|\{v \in V(G) : \deg^+(v) < d\}| \leq 2\frac{d-1}{p} + 2A\sqrt{\frac{n}{p}} + 1$$

230 *Proof.* Let G be as in the statement. We may and shall assume that C is large
231 enough so that the required inequalities hold. Throughout this proof, W denotes

232 the set of vertices with degree strictly below $(1 - 2\varepsilon/3)np$. Firstly, we prove (i). We
 233 claim that $|W| < \varepsilon n/2$. Indeed, suppose the contrary and consider a subset $W' \subseteq W$
 234 of size precisely $\varepsilon n/2$. By Fact 17, we have

$$\begin{aligned}
 235 \quad e(W') &\geq p \frac{(\varepsilon n/2)^2}{3} - A\sqrt{np}(\varepsilon n/2) = p \frac{(\varepsilon n/2)^2}{3} \left(1 - \sqrt{\frac{36A^2}{\varepsilon^2 np}}\right) \\
 236 \quad &> p \frac{(\varepsilon n/2)^2}{4} = \frac{\sqrt{2}}{16} \sqrt{\frac{\varepsilon^3 np}{1 - \varepsilon/2}} \sqrt{\frac{\varepsilon}{2}(1 - \varepsilon/2)n^3 p} \\
 237 \quad &\geq A\sqrt{np \frac{\varepsilon n}{2}(1 - \varepsilon/2)n} = A\sqrt{np|W'|(n - |W'|)}. \quad (20)
 \end{aligned}$$

238 Now, note that $|V(G) \setminus W'| < n < A^2 C n / (2\varepsilon^2 p) \leq \varepsilon n^2 p / 2 = np|W'|$, but

$$\begin{aligned}
 239 \quad e(W', V(G) \setminus W') &< |W'| \cdot (1 - 2\varepsilon/3)np - 2e(W') \\
 240 \quad &< |W'| \cdot (1 - \varepsilon/2)np - 2e(W') \\
 241 \quad &= p|W'|(n - |W'|) - 2e(W') \\
 242 \quad &\stackrel{(20)}{\leq} p|W'|(n - |W'|) - A\sqrt{np|W'|(n - |W'|)},
 \end{aligned}$$

243 which contradicts the weak bijumbledness of G .

244 Similarly, we show that the set Z of vertices having degree strictly greater
 245 than $(1 + 2\varepsilon/3)pn$ satisfies $|Z| < \varepsilon n/2$, which together with the argument above
 246 proves (i). More precisely, suppose $|Z| \geq \varepsilon n/2$, fix $Z' \subseteq Z$ with $|Z'| = \varepsilon n/2$. We
 247 claim that $A\sqrt{np|Z'|}$ and $A\sqrt{np|Z'|(n - |Z'|)}$ are both small (constant) fractions
 248 of $p|Z'|^2$. Indeed, as $|Z'|^2 < |Z'|(n - |Z'|) < |Z'|n$, it follows that

$$\begin{aligned}
 249 \quad \frac{A\sqrt{np|Z'|}}{p|Z'|^2} &< \frac{A\sqrt{np|Z'|(n - |Z'|)}}{p|Z'|^2} < \frac{A\sqrt{n^2 p|Z'|}}{p|Z'|^2} \\
 250 \quad &= \sqrt{\frac{A^2 n^2}{p|Z'|^3}} = \sqrt{\frac{A^2 n^2}{p(\varepsilon n/2)^3}} \stackrel{(\#)}{\leq} \sqrt{\frac{8p}{C}},
 \end{aligned}$$

251 where $(\#)$ is due to $\varepsilon^3 np^2 \geq CA^2$. Fact 17 and the previous inequalities imply

$$\begin{aligned}
 252 \quad e(Z') &< \frac{p|Z'|^2}{2} + A\sqrt{np|Z'|} \\
 253 \quad &< p|Z'|^2 \left(\frac{1}{2} + \sqrt{\frac{8p}{C}}\right) \\
 254 \quad &< p|Z'|^2 \left(\frac{1}{2} + \sqrt{\frac{32p}{C}}\right) - A\sqrt{np|Z'|(n - |Z'|)} \\
 255 \quad &< p|Z'|^2 - A\sqrt{np|Z'|(n - |Z'|)}.
 \end{aligned}$$

256 Analogously, we have $|V(G) \setminus Z'| < np|Z'|$, but

$$\begin{aligned}
 257 \quad e(Z', V(G) \setminus Z') &\geq (1 + 2\varepsilon/3)np|Z'| - 2e(Z') \\
 258 \quad &= p|Z'|(n - |Z'|) + \left(\frac{1}{2} + \frac{2}{3}\right)\varepsilon np|Z'| - 2e(Z') \\
 259 \quad &> p|Z'|(n - |Z'|) + 2p|Z'|^2 - 2e(Z') \\
 260 \quad &> p|Z'|(n - |Z'|) + A\sqrt{np|Z'|(n - |Z'|)},
 \end{aligned}$$

261 which is again a contradiction to Definition 15. This concludes the proof of (i).

262 To prove (ii), fix an orientation D of G and put $X = \{v \in V(G) : \deg_D^+(v) < d\}$.
 263 Fact 17 then yields the desired inequality:

$$264 \quad |X|(d-1) \geq \sum_{v \in X} \deg_{G[X]}^+(v) = e(G[X]) \geq \binom{|X|}{2} p - A\sqrt{np}|X|. \quad \square$$

265 4.1 Almost all orientations of weakly bijumbled graphs

266 In this section we show that almost every orientation of a weakly bijumbled graph
 267 contains a Seymour vertex.

Theorem 5. For every $A > 0$, there exists a constant $C = C(A) > 1$ such that
 the following holds for every $p, \varepsilon \in (0, 1)$ and $n \in \mathbb{N}$ satisfying $\varepsilon^3 np^2 \geq A^2 C$ and
 $p < 1 - 13\sqrt{\varepsilon}$. Let G be a weakly $(p, A\sqrt{np})$ -bijumbled graph of order n . If D is
 chosen uniformly at random among the $2^{e(G)}$ possible orientations of G , then D has
 a Seymour vertex with probability at least $1 - 2^{-n}$.

Proof. We may and shall assume that $A^2 C$, and hence n , is larger than any given
 268 absolute constant. Let $V = V(G)$, and fix an arbitrary orientation of G . For
 269 simplicity, we write G for both the oriented and underlying graphs. Let $\text{BAD} = \{v \in$
 270 $V(G) : \deg^+(v) < 2\sqrt{\varepsilon np}\}$. By Lemma 18 (ii), we must have

$$272 \quad |\text{BAD}| \leq \frac{2(2\sqrt{\varepsilon np} - 1)}{p} + 2A\sqrt{\frac{n}{p}} + 1 < 5\sqrt{\varepsilon n}.$$

273 Put $U = V \setminus \text{BAD}$.

Claim 19. There exists $w \in U$ such that

$$\deg_G^+(w) < n/2 - \sqrt{\varepsilon n}.$$

274 *Proof.* Recall that $p < 1 - 13\sqrt{\varepsilon}$. Hence $\varepsilon < 13^{-2} < 1$ and

$$275 \quad \frac{(1+\varepsilon)p}{2} + 5\sqrt{\varepsilon} < \frac{(1+\varepsilon)(1-13\sqrt{\varepsilon})}{2} + 5\sqrt{\varepsilon} < \frac{1-2\sqrt{\varepsilon}}{2}. \quad (21)$$

276 Note also that $\varepsilon^3 np^2 \geq A^2 C$ yields $A \leq \sqrt{\varepsilon^3 np^2 / C}$. Hence,

$$277 \quad A\sqrt{np} \leq \varepsilon np \sqrt{\frac{\varepsilon p}{C}} < \frac{\varepsilon np}{2}. \quad (22)$$

278 By Fact 17, we have

$$279 \quad \frac{e(G[U])}{|U|} \leq \frac{p}{|U|} \binom{|U|}{2} + A\sqrt{np} \leq \frac{p|U|}{2} + A\sqrt{np} \stackrel{(22)}{<} (1+\varepsilon) \frac{np}{2}. \quad (23)$$

280 Owing to (23), averaging the outdegrees of vertices in U yields that some $w \in U$
 281 satisfies $\deg_{G[U]}^+(w) \leq e(G[U])/|U| < (1+\varepsilon)np/2$. Hence,

$$282 \quad \deg_G^+(w) \leq \deg_{G[U]}^+(w) + |\text{BAD}| \\
 283 \quad < \frac{(1+\varepsilon)np}{2} + 5\sqrt{\varepsilon n} \stackrel{(21)}{\leq} \frac{(1-2\sqrt{\varepsilon})n}{2}. \quad \diamond$$

284 Note that since we picked an arbitrary orientation of G , the vertex w given by
 285 Claim 19 exists for any such orientation. To conclude the proof, we next show that
 286 in a random orientation of G almost surely every vertex in U is an $(1 - 2\sqrt{\varepsilon})$ -king,
 287 where a vertex v is said to be a λ -king if the number of vertices z for which there
 288 exists a directed path of length 2 from v to z is at least λn .

Claim 20. In a random orientation of G , the following holds with probability at least $1 - 2^{-n}$. For each $X \subseteq V(G)$ with $|X| = 2\sqrt{\varepsilon}np$, we have $|N_G^1(X)| \geq (1 - 2\sqrt{\varepsilon})n$, where $N_G^1(X) = \bigcup_{x \in X} N_G^1(x)$.

289 *Proof.* Note that for all $X, Y \subseteq V(G)$, there exist $X' \subseteq X$ and $Y' \subseteq Y$ such that
 290 $X' \cap Y' = \emptyset$ and $|X'| = |X|/2$ and $|Y'| = |Y|/2$. Fix $X \subseteq V(G)$ with $|X| = 2\sqrt{\varepsilon}np$.
 291 If we choose Y such that $|Y| = 2\sqrt{\varepsilon}n$, then $|X'| \leq |Y'| = \sqrt{\varepsilon}n \leq \sqrt{\varepsilon}n^2p^2 = np|X'|$
 292 because $np^2 \geq A^2C/\varepsilon^3 \geq 1$. Hence, as G is weakly bijumbled,

$$293 \quad e(X, Y) \geq e(X', Y') \geq \frac{p|X||Y|}{4} - \frac{A\sqrt{np|X||Y|}}{2}. \quad (24)$$

294 Let \mathcal{E}_X denote the ‘bad’ event that $|N_G^1(X)| < (1 - 2\sqrt{\varepsilon})n$, so \mathcal{E}_X occurs if and only
 295 if there exists $Y \subseteq V(G)$ with $|Y| = 2\sqrt{\varepsilon}n$ such that $\bar{e}(X, Y) = 0$. For any X such
 296 that $|X| = 2\sqrt{\varepsilon}np$, summing over all Y of size $2\sqrt{\varepsilon}n$ yields

$$297 \quad \Pr[\mathcal{E}_X] \leq \sum_Y 2^{-e(X, Y)} \stackrel{(24)}{\leq} \binom{n}{2\sqrt{\varepsilon}n} \exp(-(\ln 2)(\varepsilon n^2 p^2 - A\sqrt{\varepsilon n^3 p^2}))$$

$$298 \quad \leq \exp\left(2n\sqrt{\varepsilon} \ln\left(\frac{e}{2\sqrt{\varepsilon}}\right) - (\ln 2)\varepsilon n^2 p^2 \left(1 - \frac{\varepsilon}{\sqrt{C}}\right)\right)$$

$$299 \quad \leq \exp\left(2n\sqrt{\varepsilon} \left(\frac{e}{2\sqrt{\varepsilon}}\right) - (\ln 2)\varepsilon n^2 p^2 \left(1 - \frac{\varepsilon}{\sqrt{C}}\right)\right)$$

$$300 \quad \leq \exp(-2(\ln 2)n) \quad (25)$$

301 using that $\varepsilon np^2 \geq A^2C\varepsilon^{-2} \geq 12$ and that $\varepsilon/\sqrt{C} \leq C^{-1/2} < 1/2$ because $\varepsilon < 1$ and
 302 C is a large constant. Taking a union bound over all X of size $2\sqrt{\varepsilon}np$, we see that
 303 with high probability no bad event occurs, since

$$304 \quad \sum_X \Pr[\mathcal{E}_X] \stackrel{(25)}{\leq} 2^n \exp(-2(\ln 2)n) = 2^{-n},$$

305 and the claim holds as required. \diamond

306 To complete the proof, we now show that the vertex w given by Claim 19 is a
 307 Seymour vertex in any orientation of G that satisfies the property shown in Claim 20.
 308 By Claim 19, we have $\deg^+(w) < (1 - 2\sqrt{\varepsilon})n/2$. Moreover, since $w \notin \text{BAD}$, we have
 309 $\deg^+(w) \geq 2\sqrt{\varepsilon}np$, and hence Claim 20 implies that

$$310 \quad |N_G^1(N_G^1(w))| \geq (1 - 2\sqrt{\varepsilon})n > 2\deg^+(w).$$

311 Therefore, we have

$$312 \quad |N_G^2(w)| \geq |N_G^1(N_G^1(w))| - \deg_G^+(w) > \deg_G^+(w) = |N_G^1(w)|. \quad \square$$

5 Concluding remarks

In this paper we confirmed Seymour’s Second Neighborhood Conjecture (SNC) for a large family of graphs, including almost all orientations of (pseudo)random graphs. We also prove that this conjecture holds a.a.s. for arbitrary orientations of the random graph $G(n, p)$, where $p = p(n)$ lies below $1/4$. Interestingly, this range of p encompasses both sparse and dense random graphs.

The main arguments in our proofs lie in finding a vertex w of relatively low outdegree whose outneighborhood contains many vertices of somewhat large outdegree. Since outneighbors of w cannot have small common outneighborhood, we conclude that $|N^2(w)|$ must be large.

Naturally, it would be interesting to extend further the range of densities for which arbitrary orientations of $G(n, p)$ satisfy the SNC.

It seems likely that other classes of graphs, such as (n, d, λ) -graphs, are susceptible to attack using this approach. Theorem 4 is also a small step towards the following weaker version of Conjecture 1.

Question 21. Do most orientations of an arbitrary graph G satisfy the SNC?

Acknowledgments

The authors thank Yoshiharu Kohayakawa for useful discussions, in particular for suggesting we consider bijumbled graphs. We also thank the reviewers for helpful suggestions which greatly improved the presentation of the material.

This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil – CAPES – Finance Code 001. F. Botler is supported by CNPq (423395/2018-1) and by FAPERJ (211.305/2019 and 201.334/2022). P. Moura is supported by FAPEMIG (APQ-01040-21). T. Naia is supported by CNPq (201114/2014-3) and FAPESP (2019/04375-5, 2019/13364-7, 2020/16570-4). FAPEMIG, FAPERJ and FAPESP are, respectively, Research Foundations of Minas Gerais, Rio de Janeiro and São Paulo. CNPq is the National Council for Scientific and Technological Development of Brazil.

References

- [1] F. Botler, P. Moura, and T. Naia. Seymour’s Second Neighborhood Conjecture in arbitrary orientations of a random graph. In *Discrete Mathematics Days 2022*, volume 263, page 58. Ed. Universidad de Cantabria, 2022.
- [2] F. Botler, P. Moura, and T. Naia. Seymour’s Second Neighborhood Conjecture on sparse random graphs. In *Anais do VII Encontro de Teoria da Computação*, pages 37–40. SBC, 2022.
- [3] G. Chen, J. Shen, and R. Yuster. Second neighborhood via first neighborhood in digraphs. *Ann. Comb.*, 7(1):15–20, 2003.
- [4] Z. Cohn, A. Godbole, E. Wright Harkness, and Y. Zhang. The number of Seymour vertices in random tournaments and digraphs. *Graphs Combin.*, 32(5):1805–1816, 2016.

- 349 [5] D. Conlon and W. T. Gowers. Combinatorial theorems in sparse random sets. *Ann. of Math. (2)*,
350 184(2):367–454, 2016.
- 351 [6] N. Dean and B. Latka. Squaring the tournament—an open problem. *Congr. Numer.*, 109:73–80,
352 1995.
- 353 [7] D. Fidler and R. Yuster. Remarks on the second neighborhood problem. *J. Graph Theory*, 55(3):208–
354 220, 2007.
- 355 [8] D. Fisher. Squaring a tournament: a proof of Dean’s conjecture. *J. Graph Theory*, 23(1):43–48, 1996.
- 356 [9] S. Ghazal. Seymour’s second neighborhood conjecture for tournaments missing a generalized star.
357 *J. Graph Theory*, 71(1):89–94, 2012.
- 358 [10] F. Havet and S. Thomassé. Median orders of tournaments: a tool for the second neighborhood
359 problem and Sumner’s conjecture. *J. Graph Theory*, 35(4):244–256, 2000.
- 360 [11] P. E. Haxell, Y. Kohayakawa, and T. Łuczak. The induced size-Ramsey number of cycles. *Combin.*
361 *Probab. Comput.*, 4(3):217–239, 1995.
- 362 [12] S. Janson, T. Łuczak, and A. Ruciński. *Random graphs*. Wiley-Interscience, New York, 2000.
- 363 [13] M. Schacht. Extremal results for random discrete structures. *Ann. of Math. (2)*, 184(2):333–365,
364 2016.
- 365 [14] T. Seacrest. The arc-weighted version of the second neighborhood conjecture. *J. Graph Theory*,
366 78(3):219–228, 2015.

367 A Proof that $G(n, p)$ is p -typical (Lemma 10)

368 In this section, we show that $G(n, p)$ satisfies the standard properties of Definition 9.
369 To simplify this exposition, we make use of Lemma 22 below. Let $B \sim \mathcal{B}(N, p)$ denote
370 that B is a binomial random variable corresponding to the number of successes in N
371 mutually independent trials, each with success probability p .

Lemma 22. For all $N \in \mathbb{N}$, all $p \in (0, 1)$ and all positive x , if $B \sim \mathcal{B}(N, p)$ then

$$\Pr[|B - Np| > \sqrt{6Np(1-p)x} + 2x] < 2 \exp(-3x).$$

372 Lemma 22 follows from the following Chernoff inequality (see [12, Lemma 2.1]).

Lemma 23. Let $X \sim \mathcal{B}(N, p)$ and $\sigma^2 = Np(1-p)$. For all $t > 0$ we have

$$\Pr[|X - \mathbb{E} X| > t] < 2 \exp\left(-\frac{t^2}{2(\sigma^2 + t/3)}\right).$$

373 *Proof of Lemma 22 using Lemma 23.* Let $\sigma^2 = Np(1-p)$ and $t = \sqrt{x^2 + 6x\sigma^2} + x$.
374 Since $(t - x)^2 = x^2 + 6x\sigma^2$, we have $t^2 = 2tx + 6x\sigma^2 = 6x(\sigma^2 + t/3)$. By Lemma 23,

$$\Pr[|B - \mathbb{E} B| > t] < 2 \exp\left(-\frac{t^2}{2(\sigma^2 + t/3)}\right) = 2 \exp(-3x). \quad (26)$$

376 Since $t \leq \sqrt{6\sigma^2 x} + 2x$, we have

$$\Pr[|B - \mathbb{E} B| > \sqrt{6\sigma^2 x} + 2x] \leq \Pr[|B - \mathbb{E} B| > t] \stackrel{(26)}{<} 2 \exp(-3x). \quad \square$$

378 We next show that $G(n, p)$ is p -typical. The properties in Definition 9 follow by
379 choosing x in Lemma 22 so as to make the appropriate union bound small.

Lemma 10. For every $p: \mathbb{N} \rightarrow (0, 1)$, a.a.s. $G = G(n, p)$ is p -typical. |

Proof. We will show that a.a.s. (i)–(iv) of Definition 9 hold. Given a random variable Z and $x > 0$, let $\mathbb{1}(Z, x)$ be the indicator variable of the ‘bad’ event

$$|Z - \mathbb{E} Z| > \sqrt{6x \operatorname{Var}(Z)} + 2x,$$

where $\operatorname{Var}(Z)$ is the variance of Z . By Lemma 22, if $Z \sim \mathcal{B}(N, p)$ then

$$\mathbb{E}(\mathbb{1}(Z, x)) = \Pr[\mathbb{1}(Z, x) = 1] < 2 \exp(-3x). \quad (27)$$

Firstly, we show that a.a.s. (i) holds. For each $X \subseteq V(G)$, let $Z_X = e(X)$ and let

$$Z^* = \sum_{X \subseteq V(G)} \mathbb{1}(Z_X, n),$$

taking $x = n$. Note that $Z_X \sim \mathcal{B}(\binom{|X|}{2}, p)$ for all X . By linearity of expectation,

$$\mathbb{E} Z^* = \sum_{X \subseteq V(G)} \mathbb{E}(\mathbb{1}(Z_X, n)) \stackrel{(27)}{<} \sum_{X \subseteq V(G)} 2 \exp(-3n) = 2^{n+1} \exp(-3n) = o(1).$$

Since $Z^* \geq 0$ (it is the sum of indicator random variables), we may use Markov’s inequality, obtaining $\Pr[Z^* \geq 1] \leq \mathbb{E} Z^* = o(1)$.

A similar calculation, considering in turn $\deg(v)$ or $N(u) \cap N(v)$ instead of $e(X)$, proves that each of the items (iii) and (iv) fails to hold with probability $o(1)$, taking x as $\ln n$ in both cases, and taking union bounds over n or $\binom{n}{2}$ events respectively. Hence $G(n, p)$ satisfies properties (i), (iii) and (iv) with probability $1 - o(1)$.

The strategy to prove (ii) is similar to the above, but calculating the number of events in the union bound is slightly more involved. Note that the statement is trivially true if either X or Y is empty, so we may suppose otherwise. In particular, we may assume that $n' \geq 1$. We have two cases to consider, according to which of the hypotheses of (ii) is satisfied by n' and n'' . If $n' = n'' = n$, then (as above) we consider $e(X, Y)$ in place of $e(X)$, let $x = n$ and take a union bound over 2^{2n} events (there are in fact fewer than 2^{2n} events as we only consider disjoint sets). Otherwise, if $1 \leq n' \ln n \leq n'' \leq n$, let Ω be the set of pairs $\{X, Y\}$ of disjoint sets with $X, Y \subseteq V(G)$ and $|X|, |Y| \leq n'$, and we have $|\Omega| \leq (\sum_{i=1}^{n'} \binom{n}{i})^2$. Since $i \leq n' < n/2$ for sufficiently large n , we have $\binom{n}{i} \leq \binom{n}{n'} \leq (en/n')^{n'}$ and therefore

$$|\Omega| \leq \left(\sum_{i=1}^{n'} \binom{n}{i} \right)^2 < \left(n' \binom{n}{n'} \right)^2 < \left(n' \left(\frac{en}{n'} \right)^{n'} \right)^2 < \exp(2n'(1 + \ln n))$$

By Lemma 22, for each $\{X, Y\} \in \Omega$ we have $\Pr[\mathbb{1}(e(X, Y), n'')] < 2 \exp(-3n'')$. Applying Markov’s inequality to $Z^* = \sum_{\{X, Y\} \in \Omega} \mathbb{1}(e(X, Y), n'')$, we obtain

$$\Pr[Z^* \geq 1] \leq \mathbb{E} Z^* < \exp(2n'(1 + \ln n)) \cdot 2 \exp(-3n'') \leq 2 \exp(-n''/2) = o(1),$$

where we use that $\ln n \leq n' \ln n \leq n''$. □